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INVITED PAPER

# ON PRODUCTS OF CONJUGACY CLASSES OF THE SYMMETRIC GROUP

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The product of conjugacy classes of the symmetric group $\mathfrak{S}_n$ in its group algebra is found as a linear combination of conjugacy classes with integer coefficients. The purpose of this paper is to give a partial answer to the problem of finding simple combinatorial rules to obtain these coefficients. In particular, we will show that the product $C^{(n)} * C^{(n)}$ of the class of circular permutations with itself can be decomposed in a simple manner.

## 0. Introduction

In [4], Murnaghan gave a method for computing the value of the irreducible characters $\chi^\lambda$ (indexed by the partititions $\lambda$ of $n$) of the symmetric group $\mathfrak{S}_n$ on a conjugacy class $C_\mu$ of type $\mu = (1^{\alpha_1}2^{\alpha_2} \cdots n^{\alpha_n})$. This method provides for each $\lambda$, a closed expression $M^\lambda(\mu)$, made of binomial coefficients of the form $\binom{\alpha_i}{j}$. For example it is well known that when $\lambda = (1, n-1)$, we have $M^\lambda(\mu) = \alpha_1 - 1$. In Section 2, we use this method to obtain a closed formula for the characters $\chi^{1^r n-r}$ in hook shape (see also [6]). This formula leads us in Section 3 to a combinatorial rule for the computation of the products $C^{(n)} * C^\mu$ of the conjugacy class of circular permutations with any other class in the group algebra of $\mathfrak{S}_n$ expanded as a linear combination of conjugacy classes. As an important special case, we obtain the following combinatorial rule for the decomposition of $C^{(n)} * C^{(n)}$:

Let $\mu = (1^{\alpha_1}2^{\alpha_2} \cdots n^{\alpha_n})$ then:

$$C^{(n)} * C^{(n)} \,|_{C_\mu} = \frac{(n-1)!}{(n+1)\,|C_\mu|} \sum_{\mu' \lhd \mu} \operatorname{sgn}(\mu')\,|C_{\mu'}|\,|C_{\mu-\mu'}|$$

where $\mu' = (1^{i_1}2^{i_2} \cdots n^{i_n})$ and $\operatorname{sgn}(\mu')$ is the signature of a permutation of cycle type $\mu'$. Stanley also gave a formula ([6], Theorem 3.1) for the decomposition of $C^{(n)} * C^{(n)}$ which corresponds to our formula (12). Another noticeable instance of our general result is the following:

$$C^{(n)} * C^{(1,n-1)} \,|_{C^\mu} = \begin{cases} 2(n-2)! & \text{if } \operatorname{sgn}(\mu) = -1 \\ 0 & \text{otherwise,} \end{cases}$$

i.e. any odd permutation can be written in $2(n-2)!$ ways as a product of a circular permutation with a quasi-circular permutation. This observation made by Schützenberger [5] was the starting point of my work.

The general problem of finding the decomposition of $C^\lambda * C^\mu$ can be seen as the dual of the decomposition of the internal tensor product (Kronecker product) of two irreducible representations of $\mathfrak{S}_n$ into a linear combination of irreducible representations. Other decompositions of expressions $C^\lambda * C^\mu$ have been found by the author and the problem will be the main subject of his doctoral dissertation. Character theory was also used in [6], and our results have the same root ([6], Lemma 2.2).

The background needed for following the exposition given in this paper is shortly described in Section 1 and contained in Macdonald's monograph [3] and in the volume of James and Kerber [2].

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## 1. Preliminary results

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a *partition* of the positive integer $n$ (noted $\lambda \vdash n$) such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k$, with *length* $l(\lambda) = k$ equal to the number of non-empty parts of $\lambda$. Another useful notation for the description of a partition $\mu$ which we shall use is $\mu = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ which means that $\mu$ has $\alpha_i$ parts equal to $i$. Lets define a *partial order* $\lhd$ on the set of partitions of all numbers by saying that for $\mu' = (1^{\beta_1} 2^{\beta_2} \cdots m^{\beta_m}) \vdash m$, $\mu' \lhd \mu$ if and only if for all $i = 1, 2, \ldots, \beta_i \leqslant \alpha_i$. To each partition $\lambda$, we associate a *Ferrers diagram* having $\lambda_i$ nodes on the $i$th row, the last row being the bottom one. For example:

$$\lambda = (1, 3, 5) \quad \leftrightarrow \quad \begin{array}{l} \bullet \\ \bullet\bullet\bullet \\ \bullet\bullet\bullet\bullet\bullet \end{array}$$

If $\lambda$, $\lambda'$ are partitions, the *inclusion order* $\subset$ on the set of partitions is defined by saying that $\lambda' \subset \lambda$ if the diagram of $\lambda$ contains the diagram of $\lambda'$.

Let $\{\lambda\}$ denote the irreducible linear representation of $\mathfrak{S}_n$ associated to $\lambda$ and let $f^\lambda$ be its dimension, given by the *hook formula*

$$f^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}}, \tag{1}$$

where the $h_{i,j}$ are the lengths of of the hooks in the diagram $\lambda$ whose corner node lie in position $(i, j)$.

Let $\chi^\lambda_\mu$ be the character of $\{\lambda\}$ evaluated at the conjugacy class $C_\mu$ and let $C^\mu$ be the corresponding element of the group algebra of $\mathfrak{S}_n$ defined by:

$$C^\mu = \sum_{\sigma\in\mathfrak{S}_n} \chi(\sigma \in C_\mu)\sigma,$$

where $\chi$ is the usual characteristic function. If we also define $\chi^\lambda$ as an element of $\mathscr{C}[\mathfrak{S}_n]$, the center of the algebra of $\mathfrak{S}_n$:

$$\chi^\lambda = \sum_{\sigma\in\mathfrak{S}_n} \chi^\lambda_\sigma \sigma;$$

it is well known (see [2] or [3]) that $\{\chi^\lambda\}_{\lambda\vdash n}$ and $\{C^\mu\}_{\mu\vdash n}$ are two different basis for $\mathscr{C}[\mathfrak{S}_n]$ with transition matrix the character table; i.e.

$$C^\mu = \sum_{\lambda\vdash n} \frac{|C_\mu|}{|\mathfrak{S}_n|} \chi^\lambda_\mu \chi^\lambda \quad (\text{for all } \mu \vdash n). \tag{2}$$

The *Murnaghan–Nakayama rule* for computing $\chi^\lambda_\mu$ is a recursive method which relates $\chi^\lambda_\mu$ and $\chi^{\lambda'}_{\mu-i}$ where $\mu - i = (1^{\alpha_1}\cdots i^{\alpha_i-1}\cdots n^{\alpha_n}) \vdash n-i$ and $\lambda'$ is a subdiagram of $\lambda$ obtained by taking a *border strip* out of $\lambda$, i.e. a connected subset that lies on the north-east most part of $\lambda$ containing no $2\times 2$ block of nodes. The *length* $|s_i| = i$, of a border strip $s_i$, is the total number of nodes it contains and its *height* $h(s_i)$ is one less than the number of rows it occupies in $\lambda$. The Murnaghan–Nakayama rule states:

$$\chi^\lambda_\mu = \sum_{\lambda'\vdash n-i} (-1)^{h(\lambda-\lambda')} \chi^{\lambda'}_{\mu-i} \quad (\text{for all } \lambda,\ \mu \vdash n),$$

where the sum is taken over all $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k) \subset \lambda$ such that the diagram $\lambda - \lambda' = (\lambda_1 - \lambda'_1, \lambda_2 - \lambda'_2, \ldots, \lambda_k - \lambda'_k)$ is a border strip of length $i$.

*Murnaghan's method*

In [4], Murnaghan describes a method to obtain for each $\lambda$, a closed formula $M^\lambda(\mu)$ giving $\chi^\lambda_\mu$ in terms of the binomial coefficients $\binom{\alpha_i}{j}$. $M^\lambda(\mu)$ is a polynomial expression in the binomial coefficients $\alpha_1$, $\binom{\alpha_2}{1}$, $\binom{\alpha_2}{2}$, $\ldots$, $\binom{\alpha_3}{1}$, $\binom{\alpha_3}{2}$, $\ldots$ which is obtained as follows:

(a) the part of $M^\lambda(\mu)$ containing only $\alpha_1$ is obtained by setting $n = \alpha_1$ in the hook formula (1) for $f^\lambda$.

*Example* 1. For $\lambda = (1, 2, n-3)$, using the hook formula, we obtain:

$$f^{(1,2,n-3)} = \frac{n(n-2)(n-4)}{3} \Rightarrow M^\lambda(\mu)\,|_{\alpha_1} = f^\lambda\,|_{\alpha_1} = \frac{\alpha_1(\alpha_1-2)(\alpha_1-4)}{3}.$$

(b) The terms of $M^\lambda(\mu)$ that contain only $\alpha_1$ and $\binom{\alpha_2}{1}$ are obtained by the equality:

$$M^\lambda(\mu)\,|_{\binom{\alpha_2}{1}} = \sum_{\lambda'} (-1)^{h(\lambda-\lambda')} M^{\lambda'}(\mu)\,|_{\alpha_1},$$

where the sum is taken over all $\lambda' \subset \lambda$ such that $\lambda - \lambda'$ is a border strip of length 2 that does not intersect the bottom line $\lambda_k$ of $\lambda$.

*Example* 2. For $\lambda = (1, 1, 3, n-5)$, the coefficient of $\binom{\alpha_2}{1}$ in $M^\lambda(\mu)$ is obtained as follows:

$$M\begin{pmatrix}\bullet & & & & \\ \bullet & & & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \cdots\end{pmatrix}(\mu)\,\Big|_{\binom{\alpha_2}{1}} = (-1)f\begin{array}{l}\bullet\bullet\bullet\\ \bullet\bullet\bullet\bullet\cdots\end{array}\Big|_{\alpha_1} + f\begin{array}{l}\bullet\\ \bullet\\ \bullet\\ \bullet\bullet\bullet\bullet\cdots\end{array}\Big|_{\alpha_1}$$
$$= -\left(\binom{\alpha_1}{3} - \binom{\alpha_2}{2}\right) + \left(\binom{\alpha_1}{3} - \binom{\alpha_1}{2} + \alpha_1 - 1\right)$$
$$= \alpha_1 - 1.$$

(c) The terms of $M^\lambda(\mu)$ that contain only $\alpha_1$ and $\binom{\alpha_2}{2}$ are obtained by taking 2 border strips of length 2, out of $\lambda$ in all possible ways:

$$M^\lambda(\mu)\,\Big|_{\binom{\alpha_2}{2}} = \sum_{\lambda'}\sum_{\lambda''} (-1)^{h(\lambda-\lambda')+h(\lambda'-\lambda'')} M^{\lambda''}(\mu)\,\big|_{\alpha_1}.$$

Here the sums are taken over all $\lambda' \subset \lambda$ and $\lambda'' \subset \lambda'$ such that $\lambda - \lambda'$ and $\lambda' - \lambda''$ are border strips of length 2, both border strips not using the bottom line $\lambda_k$ of $\lambda$.

(d) In general the contribution of $\binom{\alpha_i}{j}$ which is independent of $\alpha_{i+1}, \ldots, \alpha_n$ is similarly obtained by taking successively $j$ border strips of length $i$ out of $\lambda$ and by computing the contribution of $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$, in the remaining diagrams:

$$M^\lambda(\mu)\,\Big|_{\binom{\alpha_i}{j}} = \sum_{(\lambda^1,\lambda^2,\ldots,\lambda^j)} \prod_{k=0}^{j-1} (-1)^{h(\lambda^k-\lambda^{k+1})} M^{\lambda^j}(\mu)\,\big|_{\alpha_1,\alpha_2,\ldots,\alpha_{i-1}}.$$

Here the sum is taken over all $j$-tuples $(\lambda^1, \lambda^2, \ldots, \lambda^j)$ of partitions with $\lambda = \lambda^0 \supset \lambda^1 \supset \lambda^2 \cdots \supset \lambda^j$, each $\lambda^k - \lambda^{k+1}$ being a border strip of length $i$, none of the border strips using $\lambda_k$.

*Example* 3. For $\lambda = (1, 1, 3, n-5)$, the different components of $M^\lambda(\mu)$ are obtained as follows;

$$f^\lambda\,\big|_{\alpha_1} = \frac{\alpha_1(\alpha_1-1)(\alpha_1-3)(\alpha_1-4)(\alpha_1-7)}{20},$$
$$M^\lambda(\mu)\,\Big|_{\binom{\alpha_2}{1}} = f^{(1^3,n-3)}\,\big|_{\alpha_1} - f^{(3,n-3)}\,\big|_{\alpha_1}$$
$$= \alpha_1 - 1 \quad \text{(see Example 2)},$$
$$M^\lambda(\mu)\,\Big|_{\binom{\alpha_2}{2}} = -2f^{(1,n-1)}\,\big|_{\alpha_1}$$
$$= -2(\alpha_1 - 1),$$
$$M^\lambda(\mu)\,\Big|_{\binom{\alpha_5}{1}} = M^{(n)}(\mu)\,\big|_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = 1 \Rightarrow$$
$$M^\lambda(\mu) = \chi^\lambda_\mu = \frac{\alpha_1(\alpha_1-1)(\alpha_1-3)(\alpha_1-4)(\alpha_1-7)}{20}$$
$$+ (\alpha_1 - 1)\binom{\alpha_2}{1} - 2(\alpha_1 - 1)\binom{\alpha_2}{2} + \alpha_5.$$

## 2. A formula for $\chi_\mu^{1^r,n-r}$

Using Murnaghan's method, we can now compute hook characters for $\mu=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$. First of all, it is important to observe that:

$$\chi_{1^n}^{1^r,n-r}=M^{1^r,n-r}(\mu)\mid_{\alpha_1}=f^{1^r,n-r}\mid_{\alpha_1}=\binom{\alpha_1-1}{r}, \tag{3}$$

so that we have;

$$\chi_\mu^{1^2,n-2}=M^{1^2,n-2}(\mu)\mid_{\alpha_1}-\alpha_2=\binom{\alpha_1-1}{2}-\alpha_2,$$

$$\chi_\mu^{1^3,n-3}=M^{1^3,n-3}(\mu)\mid_{\alpha_1}-\alpha_2M^{1,n-1}(\mu)\mid_{\alpha_1}+\alpha_3$$
$$=\binom{\alpha_1-1}{3}-\binom{\alpha_1-1}{1}\alpha_2+\alpha_3$$

and in general:

$$\chi_\mu^{1^r,n-r}=M^{1^r,n-r}(\mu)\mid_{\alpha_1}-\binom{\alpha_2}{1}M^{r-2,n-(r-2)}(\mu)\mid_{\alpha_1}+\binom{\alpha_3}{1}M^{1^{r-3},n-(r-3)}(\mu)\mid_{\alpha_1,\alpha_2}$$
$$+\cdots+\binom{\alpha_2}{2}M^{1^{r-4},n-(r-4)}(\mu)\mid_{\alpha_1}+\cdots$$
$$=\binom{\alpha_1-1}{r}-\binom{\alpha_2}{1}\binom{\alpha_1-1}{r-2}+\binom{\alpha_3}{1}\left[\binom{\alpha_1-1}{r-3}-\binom{\alpha_2}{1}\binom{\alpha_1-1}{r-5}+\cdots\right]$$
$$+\binom{\alpha_2}{2}\binom{\alpha_1-1}{r-4}+\cdots \tag{4}$$

thus using the binomial identity:

$$\binom{\alpha_1-1}{k}=\sum_{i=0}^{k}(-1)^i\binom{\alpha_1}{k-i}, \tag{5}$$

equality (4) can be transformed into:

$$\chi_\mu^{1^r,n-r}=(-1)^r\left[1-\sum_{i\leq r}\binom{\alpha_i}{1}+\sum_{2i\leq r}\binom{\alpha_i}{2}\right.$$
$$\left.+\cdots+\sum_{\substack{\mu'=(1^{i_1},2^{i_2},\ldots,n^{i_n})\vdash t\\ t\leq r}}(-1)^{l(\mu')}\binom{\alpha_1}{i_1}\binom{\alpha_2}{i_2}\cdots\binom{\alpha_n}{i_n}+\cdots\right], \tag{6}$$

and identities (4) or (6) are easily transformed into the following fundamental result (see also [6], Lemma 2.2):

$$\chi_\mu^{1^r,n-r}=(-1)^r\sum_{\mu'\vdash r}(-1)^{l(\mu')}\binom{\alpha_1-1}{i_1}\binom{\alpha_2}{i_2}\cdots\binom{\alpha_n}{i_n}, \tag{7}$$

where the sum is taken over all $\mu' = (1^{i_1}2^{i_2} \cdots n^{i_2})$. A shorter version of (7) follows in the same manner when the numbers $1, 2, \ldots, n_1 - 1$ do not appear in $\mu$. In this case we obtain:

**Theorem 1.** *Let $\mu = (n_1^{j_1} n_2^{j_2} \cdots n_s^{j_s})$ with $0 < n_1 < n_2 < \cdots < n_s \leq n$ and $j_1 \geq 1$ and let $r < n$ be a positive integer, then:*

$$\chi_\mu^{1^r, n-r} = (-1)^r \sum_{\substack{\mu' \vdash t \\ r - n_1 < t \leq r}} (-1)^{l(\mu')} \binom{j_1 - 1}{i_1} \binom{j_2}{i_1} \cdots \binom{j_2}{i_s} \tag{8}$$

*where the sum is taken over all $\mu' = (n_1^{i_1} n_2^{i_2} \cdots n_s^{i_s})$ that are partitions of a number $t$ satisfying $r - n_1 < t \leq r$. Observe that the fact $\mu' \lhd \mu$ is a consequence of* (8).

**Proof.** If we replace $\alpha_1, \alpha_2, \ldots, \alpha_{n_1 - 1}$ by zero in (6) and factorize $\binom{\alpha_2}{i_2} \cdots \binom{\alpha_s}{i_s}$, we get the desired result. □

*Example* 4. The value of the character

$$\chi_{2^2 3^2 4}^{1^6 8} = (-1)^6 \left[ (-1)^2 \binom{2}{2} + (-1)^2 \binom{1}{1}\binom{1}{1} + (-1)^2 \binom{1}{1}\binom{2}{1} \right] = 4$$

is directly obtained from Theorem 1 since we have to take partitions of 5 and 6 that are using the building blocks of the partition $(2^2 3^2 4)$.

It is also worth noticing that when $n = k\alpha_k$, we get from (6)

$$\chi_{k^{\alpha_k}}^{1^r, n-r} = (-1)^r \left[ 1 + \binom{\alpha_k}{1} + \binom{\alpha_k}{2} + \cdots + (-1)^i \binom{\alpha_k}{i} \right],$$

where $r - k < ik \leq r$, thus we obtain the following generalization of (3);

**Proposition 1.** *Let $n = k\alpha_k$, then for $r < n$ we have:*

$$\chi_{k^{\alpha_k}}^{1^r n-r} = (-1)^{r + \lfloor r/k \rfloor} \binom{\alpha_k - 1}{\lfloor r/k \rfloor}$$

*where $\lfloor r/k \rfloor$ is the integer part of $r/k$.*

## 3. Product of conjugacy classes

Using identity (2) and the fact that irreducible characters are primitive idempotents in $\mathscr{C}[\mathfrak{S}_n]$, i.e.

$$\chi^\lambda * \chi^\mu = \delta_{\lambda\mu} \frac{|\mathfrak{S}_n|}{f^\lambda} \chi^\lambda \quad \text{(for all } \lambda \vdash n,\ \mu \vdash n),$$

we easily derive the identity:

$$C^{\lambda} * C^{\mu} \mid_{C^{\alpha}} = \frac{|C_{\lambda}|\,|C_{\mu}|}{|\mathfrak{S}_n|} \sum_{\gamma \vdash n} \frac{\chi^{\gamma}_{\lambda}\chi^{\gamma}_{\mu}\chi^{\gamma}_{\alpha}}{f^{\gamma}}, \tag{9}$$

which is the classical way of computing the multiplicity of $C^{\alpha}$ in the product $C^{\lambda} * C^{\mu}$ in $\mathscr{C}[\mathfrak{S}_n]$. Using (7) and the fact that

$$\chi^{\lambda}_{(n)} = \begin{cases} (-1)^r & \text{if } \lambda = (1^r, n-r) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

(see [2], p. 54), we deduce the following:

**Theorem 2.** *Let $\mu_1 = (1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$ and $\mu_2 = (1^{\beta_1}2^{\beta_2}\cdots n^{\beta_n})$ be partitions of $n$, then:*

$$C^{(n)} * C^{\mu_1} \mid_{C^{\mu_2}} = \frac{|C^{\mu_1}|}{n} \sum_{r=0}^{n-1} \frac{1}{\binom{n-1}{r}} \left[ \sum_{\mu_1' \vdash r} (-1)^{l(\mu_1')} \binom{\alpha_1 - 1}{i_1}\binom{\alpha_2}{i_2}\cdots\binom{\alpha_n}{i_n} \right]$$
$$\times \left[ \sum_{\mu_2' \vdash r} (-1)^{l(\mu_2')} \binom{\beta_1 - 1}{j_1}\binom{\beta_2}{j_2}\cdots\binom{\beta_n}{j_n} \right],$$

*where $\mu_1' = (1^{i_1}2^{i_2}\cdots n^{i_n})$, $\mu_2' = (1^{j_1}2^{j_2}\cdots n^{j_n})$.*

**Proof.** We need only to observe, using (9), (10) and (3) that:

$$C^{(n)} * C^{\mu_1} \mid_{C^{\mu_2}} = \frac{(n-1)!\,|C^{\mu_1}|}{n!} \sum_{r=0}^{n-1} \frac{(-1)^r}{\binom{n-1}{r}} \chi^{1^r, n-r}_{\mu_1} \chi^{1^r, n-r}_{\mu_2}, \tag{11}$$

and then substitute (7) in this last identity. □

**Remark.** If, instead of (7), we substitute (8) into (11), we get an identity similar to Theorem 2 which gives rise to faster computation.

*Example* 5. As an illustration of this fact, we obtain:

$$C^{(10)} * C^{(3^2 4)} \mid_{C^{(2^2 3^2)}}$$
$$= \frac{2(10)!}{720} \left[ \frac{1}{\binom{9}{0}} - \frac{(-1)(-1)}{\binom{9}{1}} + \frac{(1)(-1)}{\binom{9}{2}} - \frac{(1)(3)}{\binom{9}{3}} + \frac{(-2)(-2)}{\binom{9}{4}} \right] = 8640$$

which means that any permutation of cycle type $(2^2 3^2)$ can be written in 8640 different ways as a product of a circular permutation and a permutation of type $(3^2 4)$.

We will devote the remainder of this paper to special cases of Theorem 2 where the expression can be simplified.

*Case* 1. $\mu_1 = (n)$. In this case, if we use the identity $\mathrm{sgn}(\mu) = (-1)^{n+l(\mu)}$ for $\mu \vdash n$, Theorem 2 becomes:

$$C^{(n)} * C^{(n)} \mid_{C^{\mu_2}} = \frac{(n-1)!}{n} \sum_{r=0}^{n-1} \frac{1}{\binom{n-1}{r}} \left[ \sum_{\mu_2' \vdash r} \mathrm{sgn}(\mu_2') \binom{\beta_1 - 1}{j_1} \binom{\beta_2}{j_2} \cdots \binom{\beta_n}{j_n} \right]. \quad (12)$$

Thus using the following identity (see [1], #4.1):

$$\frac{(-1)^r}{\binom{n-1}{r}} = \frac{n}{n+1} \left[ \frac{(-1)^r}{\binom{n}{r}} + \frac{(-1)^r}{\binom{n}{r+1}} \right],$$

we derive from Theorem 2 the next identity:

$$C^{(n)} * C^{(n)} \mid_{C^{\mu}} = \frac{(n-1)!}{n+1} \sum_{r=0}^{n} \frac{1}{\binom{n}{r}} \sum_{\mu' \vdash r} \mathrm{sgn}(\mu') \binom{\alpha_1}{i_1} \binom{\alpha_2}{i_2} \cdots \binom{\alpha_n}{i_n} \quad (13)$$

where $\mu = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$, $\mu' = 1^{i_1} 2^{i_2} \cdots n^{i_n}$ and $\mu' \lhd \mu$.

**Remark.** When we consider that the $r$th and $(n-r)$th terms are the same for even permutations and cancel each other for odd permutations in (13), we can either cut in half the number of steps or observe that the coefficient of conjugacy classes of odd permutations is zero.

*Example* 6. Thus we have;

$$C^{(12)} * C^{(12)} \mid_{C^{(2,3^2,4)}}$$
$$= \frac{11!}{13} \left( 2 \left[ \frac{1}{\binom{12}{0}} - \frac{1}{\binom{12}{2}} + \frac{2}{\binom{12}{3}} - \frac{1}{\binom{12}{4}} - \frac{2}{\binom{12}{5}} \right] + \frac{2}{\binom{12}{6}} \right) = 6{,}082{,}560$$

Here (13) implies that the numerator corresponding to the denominator $\binom{n}{r}$ is obtained from all $\mu' \lhd \mu$ with $\mu' \vdash r$.

A concise form for $C^{(n)} * C^{(n)}$ is obtained if we multiply the right hand side of (13) by

$$\frac{1^{\alpha_1} \cdots n^{\alpha_n}}{1^{i_1 + (\alpha_1 - i_1)} \cdots n^{i_n + (\alpha_n - i_n)}}.$$

As a matter of fact we obtain:

**Corollary 1.** *Let* $\mu = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$, *then:*

$$C^{(n)} * C^{(n)} \mid_{C_\mu} = \frac{(n-1)!}{(n+1)\,|C_\mu|} \sum_{\mu' \lhd \mu} \mathrm{sgn}(\mu')\, |C_{\mu'}|\, |C_{\mu-\mu'}|. \quad (14)$$

*Case* 2. $\mu_2 = (k, n-k)$ (with $k < n-k$). Using (8), we observe:

$$\chi^{1^r n-r}_{(k,n-k)} = \begin{cases} (-1)^r & \text{if } 0 \leqslant r < k \\ 0 & \text{if } k \leqslant r < n-k \\ (-1)^{r-1} & \text{if } n-k \leqslant r < n, \end{cases}$$

so that using the same argument leading to Theorem 2, we obtain the identity:

$$C^{(n)} * C^{(k,n-k)} \,|_{C^\mu} = \frac{2n!}{nk(n-k)} \sum_{r=0}^{k-1} \frac{(-1)^r}{\binom{n-1}{r}} \left[ \sum_{\mu' \vdash r} (-1)^{l(\mu')} \binom{\alpha_1 - 1}{i_1} \binom{\alpha_2}{i_2} \cdots \binom{\alpha_n}{i_n} \right],$$

where $\mu = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ is an odd permutation and $\mu' = 1^{i_1} 2^{i_2} \cdots n^{i_n}$. It follows that:

**Corollary 2.** *For every* $\mu = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$, *we have:*

$$C^{(n)} * C^{(k,n-k)} \,|_{C^\mu} = \frac{(1 - \operatorname{sgn}(\mu))n!}{(n+1)k(n-k)} \times \left[ \sum_{r=0}^{k-1} \frac{1}{\binom{n}{r}} \left[ \sum_{\mu' \vdash r} \operatorname{sgn}(\mu') \binom{\alpha_1}{i_1} \binom{\alpha_2}{i_2} \cdots \binom{\alpha_n}{i_n} \right] + \frac{1}{\binom{n}{k}} \sum_{\mu' \vdash k-1} \operatorname{sgn}(\mu') \binom{\alpha_1 - 1}{i_1} \binom{\alpha_2}{i_2} \cdots \binom{\alpha_n}{i_n} \right]$$

*where* $\mu' = 1^{i_1} 2^{i_2} \cdots n^{i_n}$.

**Remark.** A formula for $C^{(n)} * C^{\prime k n-k}$ can be obtained from Theorem 2, the construction of which is left to the reader.

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